RATIONAL FUNCTIONS

Definition

A function f is rational if $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials. In other words, a rational function is a ratio of two polynomials in which the denominator is not zero.

Rational Functions and their Domain

The domain of $f(x)$ consists of all real numbers except the zeros of the denominator $h(x)$. In other words, the domain is the set of all real numbers that can be used in place of the variable.

Steps;

1. Equate the denominator, $h(x)$ to zero and solve for the value of x.
2. The value of x (or the involving variable) obtained is not part of the domain if it makes the denominator zero.

In a rational function, the denominator cannot be equal to zero because that would be undefined. Find which numbers that make the fraction undefined by creating an equation whose denominator is not equal to zero.

Worked Examples

Determine the domain of the following:

1. $f(x) = \frac{4x+5}{3x-8}$
2. $f(x) = \frac{x+2}{2x+1}$
3. $f(x) = \frac{2x-7}{9x+1}$
4. $f(x) = \frac{x+5}{x^2-4}$

Solution

1. $\frac{4x+5}{3x-8}$

Let $3x - 8 = 0$

$3x = 8$

$x = \frac{8}{3}$

Domain = {x is real except $x = \frac{8}{3}$}

2. $f(x) = \frac{x+2}{2x+1}$

Let $2x + 1 = 0$

$2x = -1$

$x = -\frac{1}{2}$

3. $f(x) = \frac{2x-7}{9x+1}$

Let $9x + 1 = 0$

$9x = -1$

$x = -\frac{1}{9}$

Domain = {x is real except $x \neq -\frac{1}{9}$ }

4. $f(x) = \frac{x+5}{x^2-4}$

Let $x^2 - 4 = 0$

$x^2 - 2^2 = (x + 2)(x - 2) = 0$

$\Rightarrow x = 2$ or $x = -2$

Domain = {x is real except $x = 2$ or $x = -2$)

The Range

The range of a rational function is the same as the domain of the inverse function. Hence to find the range:

1. Find the inverse of the function.
2. Find the domain of the inverse function.

Worked Examples

Determine the range of the following:

1. $f(x) = \frac{4x+5}{3x-8}$
2. $f(x) = \frac{x+5}{x^2-4}$

Solutions

$f(x) = \frac{4x+5}{3x-8}$

$f(x) = y$

$y = \frac{4x+5}{3x-8}$

$y(3x - 8) = 4x + 5$ (*change subject to x)*

$3xy - 8y = 4x + 5$

$3xy - 4x = 8y + 5$

$x(3y - 4) = 8y + 5$

$x = \frac{8y + 5}{3y - 4}$

Now, let $3y - 4 = 0$

$3y = 4$

$y = \frac{4}{3}$

Range = **{** $y : y \in R, y \neq \frac{4}{3}$ **}**

4. $f(x) = \frac{x + 2}{x^2 - 4}$

Let $y = \frac{x + 2}{x^2 - 4}$ (*Solve for x)*

$y = \frac{x + 2}{(x+2)(x-2)}$

$y = \frac{1}{x - 2}$ (*Change subject to y)*

$y(x - 2) = 1$

$xy - 2y = 1$

$xy = 2y + 1$

$y = \frac{2y+1}{y}$

Range = $\{y : y \in R, y \neq 0\}$

Exercises 6.1

A. Find the domain of the following rational functions.

1. $f(x) = \frac{x + 4}{(x+2)(x-3)}$

2. $f(x) = \frac{x+2}{x^2-4}$

3. $f(x) = \frac{6x^2+x-1}{4x^2+x-3}$

4. $f(x) = \frac{4}{x + 5} + 5$

5. $f(x) = -\frac{x-1}{2x - 3}$

6. $f(x) = \sqrt{\frac{x-3}{1- x^2}}$

B. Find the range.

1. $f(x) = \frac{2x-6}{x+1}$

2. $f(x) = \frac{6x+12}{x - 4}$

3. $f(x) = \frac{- x+8}{x-4}$

4. $f(x) = \frac{3x+12}{x-12} + 5$

5. $f(x) = \frac{-3x+4}{x-2}$

6. $f(x) = \frac{-9x}{3x-2}$

Evaluating Rational Functions

A rational function involving a variable has no value unless a value is assigned to the variable. Once the variable is given a value, the rational function can be evaluated.

Worked Examples

1. Find the value of $f(x) = \frac{4x-1}{x+2}$ for $x = -3$

Solution

$f(x) = \frac{4x-1}{x+2}$

When $x = -3$

$f(x) = \frac{4(-3) -1}{(-3) + 2} = \frac{-13}{-1} = 13$

2. If $R(x) = \frac{3x+2}{2x - 1}$, find R(4).

Solution

$R(x) = \frac{3x+2}{2x-1}$

$R(4) = \frac{3(4)+2}{2(4)-1} = 2$

Exercises 6.2

A. Evaluate each of the following.

1. $f(x) = \frac{-3x-3}{x+5}$ for $x = -2$

2. $f(x) = \frac{3x+1}{4x-4}$ for $x = 5$

3. $R(x) = \frac{2x+9}{x}$, find R(3)

4. $f(x) = \frac{20x-2}{x-8}$, find R(-1)

B. 1. If $R(x) = \frac{x^2-2x-3}{x-2}$, find $R(3)$, $R(5)$, $R(2.05)$ and $R(1.999)$

2. Given that f(x) = $\frac{x-5}{x+3}$, find f(2), f(- 4), f(-3.02), f(-2.96)

Reducing Rational Functions

The steps involved in reducing rational functions are;

1. Factor the numerator and denominator completely.
2. Divide the numerator and denominator by the greatest common factor.

Worked Examples

1. Simplify $f(x) = \frac{x^2-9}{6x+18}$

Solution

$f(x) = \frac{x^2-9}{6x+18} = \frac{x^2-3^2}{6(x+3)} = \frac{(x+3)(x-3)}{6(x+3)} = \frac{x-3}{6}$

2. Simplify $f(w) = \frac{-3w-3w^2}{w^2-1}$

Solution

$f(w) = \frac{-3w-3w^2}{w^2-1} = \frac{-3w(1+w)}{(w+1)(w-1)} = \frac{-3w}{w-1}$

Involving Rational Functions of Two Variables

Worked Examples

1. Simplify $f(xy) = \frac{5x-5y}{4y-4x}$

Solution

$f(xy) = \frac{5x-5y}{4y-4x} = \frac{5\,(x-y)}{4(y-x)}$

But $\frac{x-y}{y-x}$ = - 1

$f(xy) = \frac{5}{4}$ (-1)

$f(xy) = -\frac{5}{4}$

2. Simplify $f(mn) = \frac{\boldsymbol{m^2-n^2}}{\boldsymbol{n-m}}$

Solution

$f(mn) = \frac{m^2-n^2}{n-m} = \frac{(m+n)(m-n)}{(n-m)}$

But $\frac{m-n}{n-m}$ = -1

$f(x)$ = (m + n) (-1)

$f(x)$ = - m – n

Exercises 6.3

Reduce each of the following.

1. $f(x) = \frac{x^2-1}{x^2+2a+1}$
2. $f(x) = \frac{3x^2+18x+27}{21x+63}$
3. $f(x) = \frac{x^3-3x^2-4x}{x^2-4x}$
4. $f(x) = \frac{2x^2+10x+12}{3x^2-27}$
5. $f(x) = \frac{2x^2+4x+2}{4x^2-4}$
6. $f(xy) = \frac{x^2-y^2}{x^2+2xy+y^2}$

Identity

If two rational functions are equivalent, then they have the same numerical value for any replacement of the variable, conditioned not to give a zero denominator. For example, given that $f(x) = \frac{x^2-9}{6x+18}$ and g(x) = $\frac{x-3}{6}$. Since $f(x)$ = $g(x)$, it implies that $\frac{x^2-9}{6x+18} = \frac{x-3}{6}$. This is satisfied by all real numbers except -3. The equation is therefore an identity.

Multiplication of Rational Functions

Rational functions are multiplied the same way as rational numbers and rational expressions. With rational functions, we can factor, reduce and then multiply.

Worked Examples

1. If $f(x) = \frac{x^2+7x+12}{2x+6}$ and $g(x) = \frac{x}{x^2-16}$, find the product of $f(x)$ and g(x).

Solution

$f(x) = \frac{x^2+7x+12}{2x+6}$ and $g(x) = \frac{x}{x^2-16}$,

$f(x) \cdot g(x) = \frac{x^2+7x+12}{2x+6} \cdot \frac{x}{x^2-16}$,

$= \frac{(x+3)(x+4)}{2(x+3)} \cdot \frac{x}{x^2-4^2}$,

$= \frac{(x+3)(x+4)}{2(x+3)} \cdot \frac{x}{(x+4)(x-4)}$,

$= \frac{1}{2} \cdot \frac{x}{x-4}$,

$= \frac{x}{2(x-4)}$

$= \frac{x}{2x-8}$

2. Given that $f(xy) = \frac{2x-2y}{4}$ and $g(xy) = \frac{2x}{x^2-y^2}$, find $f(xy) \cdot g(xy)$

Solution

$f(xy) = \frac{2x-2y}{4}$ and $g(xy) = \frac{2x}{x^2-y^2}$,

$f(xy) \cdot g(xy) = \frac{2x-2y}{4} \cdot \frac{2x}{x^2-y^2}$,

$= \frac{2(x-y)}{4} \cdot \frac{2x}{(x+y)(x-y)}$,

$= \frac{2}{4} \cdot \frac{2x}{x+y}$,

$= \frac{x}{y}$

Exercises 6.4

In each of the following, find the product of f(x) and g(x)

1. $f(x) = x^2 - 6x + 9$ and $g(x) = \frac{3}{x-3}$

2. $f(x) = \frac{12}{4x+10}$ and $g(x) = 4x^2 + 20x + 25$

3. $f(x) = \frac{6x-18}{2x^2-5x-3}$ and $g(x) = \frac{4x^2+4x+1}{6x+3}$

4. $f(x) = \frac{x^2-x-2}{x^2+5x+6}$ and $g(x) = \frac{x+2}{x^2+2x-8}$

5. $f(x) = \frac{35x^2-25x}{x^2-36}$ and $g(x) = \frac{6-x}{15x^4}$

Division of Rational Functions

Division of rational functions is similar to that of rational numbers. Invert the divisor and multiply, not forgetting to factor and reduce where possible.

Worked Examples

Given that $f(x) = \frac{4-x^2}{x^2+x}$ and $g(x) = \frac{x-2}{x^2-1}$

Find $f(x) \div g(x)$.

Solution

$f(x) = \frac{4-x^2}{x^2+x}$ and $g(x) = \frac{x-2}{x^2-1}$

$f(x) \div g(x) = \frac{4-x^2}{x^2+x} \div \frac{x-2}{x^2-1}$

$= \frac{4-x^2}{x^2+x} \times \frac{x^2-1}{x-2}$

$= \frac{2^2-x^2}{x(x+1)} \times \frac{x^2-1^2}{x-2}$

$= \frac{(2+x)(2-x)}{x(x+1)} \times \frac{(x+1)(x-1)}{x-2}$

$= \frac{(2+x)(-1)}{x} \times \frac{(x-1)}{1}$

$= \frac{-1(2+x)(x-1)}{x}$

$= \frac{-(x^2+x-2)}{x}$

$= \frac{-x^2-x+2}{x}$

Exercise 6.5

In each of the following, find $\frac{f(x)}{g(x)}$

1. $f(x) = \frac{x^2+4x+4}{8}$ and $g(x) = \frac{(x+2)^3}{16}$

2. $f(x) = \frac{x^2+2x+1}{3}$ and $g(x) = \frac{x^2-1}{x}$

3. $f(x) = \frac{x^2-7x+12}{x^2-4x}$ and $g(x) = x^2 - 9$

4. $f(x) = 2x^2 - 3x - 5$ and $g(x) = \frac{2x-5}{x-1}$

5. $f(x) = 6x^2 - x - 2$ and $g(x) = \frac{2x+1}{3x-2}$

Addition and Subtraction of Rational Function

Type 1 (Identical Denominators)

Add or subtract numerators whilst a common denominator is maintained. Simplify where possible.

Worked Examples

1. Given that $f(x) = \frac{2x}{x+2}$ and $g(x) = \frac{4}{x+2}$, find : $f(x) + g(x)$

Solution

$f(x) = \frac{2x}{x+2}$ and $g(x) = \frac{4}{x+2}$,

$f(x) + g(x) = \frac{2x}{x+2} + \frac{4}{x+2} = \frac{2x+4}{x+2} = \frac{2(x+2)}{x+2} = 2$

2. If $f(x) = \frac{x^2+2x}{(x-1)(x+3)}$ and $g(x) = \frac{2x+1}{(x-1)(x+3)}$, find $f(x) - g(x)$

Solution

$f(x) = \frac{x^2+2x}{(x-1)(x+3)}$ and $g(x) = \frac{2x+1}{(x-1)(x+3)}$

$f(x) - g(x) = \frac{x^2+2x}{(x-1)(x+3)} - \frac{2x+1}{(x-1)(x+3)}$

$= \frac{x^2+2x-(2x+1)}{(x-1)(x+3)}$

$= \frac{x^2+2x-2x-1}{(x-1)(x+3)}$

$= \frac{x^2-1}{(x-1)(x+3)}$

$= \frac{(x+1)(x-1)}{(x-1)(x+3)}$

$= \frac{x+1}{x+3}$

Type 2 (Different Denominators)

Add or subtract numerators whilst the lowest common denominator is maintained. Simplify where possible.

Worked Examples

1. Given that $f(x) = \frac{1}{x^2-9}$ and $g(x) = \frac{2}{x^2+3x}$, find $f(x) + g(x)$

Solution

$f(x) = \frac{1}{x^2-9}$ and $g(x) = \frac{2}{x^2+3x}$,

$f(x) + g(x) = \frac{1}{x^2-9} + \frac{2}{x^2+3x}$,

$= \frac{1}{x^2-3^2} + \frac{2}{x(x+3)}$,

$= \frac{1}{(x+3)(x-3)} + \frac{2}{x(x+3)}$,

$= \frac{x+2(x-3)}{x(x+3)(x-3)}$

$= \frac{x+2x-6}{x(x+3)(x-3)}$

$= \frac{3x-6}{x(x+3)(x-3)}$

2. Given that $f(x) = \frac{4}{x-a}$ and $g(x) = \frac{2}{x-5}$, find: $f(x) - g(x)$

Solution

$f(x) = \frac{4}{5-x}$ and $g(x) = \frac{2}{x-5}$,

$f(x) - g(x) = \frac{4}{5-x} - \frac{2}{x-5}$,

$= \frac{4(-1)}{5-x(-1)} - \frac{2}{x-5}$,

$= \frac{-4}{x-5} - \frac{2}{x-5}$,

$= \frac{-4-2}{x-5}$

$= \frac{-6}{x-5}$

Exercise 6.6

A. Given that $f(x) = \frac{x+1}{x^2-x-6}$,

$$g(x) = \frac{2x+3}{x^3+8x^2+20x+6},$$

$$h(x) = \frac{2x+3}{x^2+4x+4}.$$

Simplify the following, leaving your answer in factors;

1. $f(x) + g(x)$
2. $f(x) + h(x)$
3. $g(x) + h(x)$
4. $g(x) - f(x)$
5. $h(x) - f(x)$
6. $h(x) - g(x)$

B. Find the domain and zeros of the functions defined in questions 1 to 6 of *A*

Partial Fractions

Consider the identity below:

$\frac{2}{x^2-1} = \frac{1}{x-1} + \frac{-1}{x+1}$

By adding the fractions $\frac{1}{x-1}$ and $\frac{-1}{x+1}$, we obtain $\frac{2}{x^2-1}$. The expression on the R.H.S. of the equation is called the ***partial fraction decomposition of*** $\frac{2}{x^2-1}$.

For partial fraction decomposition of $\frac{f(x)}{g(x)}$ to be found, it is essential that $f(x)$ have lower degree than $g(x)$. If this is not the case, use long division to obtain such expression. For example, given $\frac{x^3-6x^2+5x-3}{x^2-1}$, we obtain $\frac{x^3-6x^2+5x-3}{x^2-1} = x - 6 + \frac{6x-9}{x^2-1}$

Type I: A Partial fraction decomposition in which each denominator is linear

A rational expression with a linear denominator is decomposed on the R.H.S. as shown below.

1. $\frac{Numerator}{x} = \frac{A}{x}$
2. $\frac{Numerator}{x(x+e)} = \frac{A}{x} + \frac{B}{x+e}$
3. $\frac{Numerator}{x(x+e)(x+f)} = \frac{A}{x} + \frac{B}{x+e} + \frac{C}{x+f}$
4. $\frac{Numerator}{x(x+e)(x+f)(x+g)} = \frac{A}{x} + \frac{B}{x+e} + \frac{C}{x+f} + \frac{D}{x+g}$,
5. $\frac{Numerator}{(x+e)(x+f)(x+g)} = \frac{A}{x+e} + \frac{B}{x+f} + \frac{C}{x+g}$

where the numerators at the R.H.S. are constants to be found.

Worked Examples

Find the partial fraction decomposition of the following:

1. $\frac{8x-1}{(x-2)(x+3)}$
2. $\frac{3x^2-16}{x^2-4x}$
3. $\frac{11x+12}{(2x+3)(x+2)(x-3)}$
4. $\frac{4x^2+13x-9}{x^3-2x^2-3x}$

Solutions

1. $\frac{8x-1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

Multiply through by the least common denominator, $(x-2)(x+3)$.

$\frac{8x-1}{(x-2)(x+3)} \cdot (x-2)(x+3) = \frac{A}{x-2} \cdot (x-2)(x+3)$

$+ \frac{B}{x+3} \cdot (x-2)(x+3)$

$\Rightarrow 8x - 1 = A(x+3) + B(x-2)$

When $x = -3$

$8(-3) - 1 = A(-3+3) + B(-3-2)$

$-24 - 1 = 0 + (-5B)$

$-25 = -5B$

$B = 5$

When $x = 2$

$8(2) - 1 = A(2+3) + B(2-2)$

$16 - 1 = 5A + 0$

$15 = 5A$

$A = 3$ $B = -2$

Now Put $A = 3$ and $B = 5$ in

$\frac{8x-1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

$\Rightarrow \frac{8x-1}{(x-2)(x+3)} = \frac{3}{x-2} + \frac{5}{x+3}$

2. $\frac{3x^2-16}{x^2-4x}$

$\Rightarrow \frac{3x^2-16}{x(x-4)} = \frac{A}{x} + \frac{B}{(x-4)}$ (*Multiply through by lcd*)

$3x^2 - 16 = A(x - 4) + Bx$

When $x = 4$,

$3(4)^2 - 16 = A(4 - 4) + 4B$

$48 - 16 = 4B$

$32 = 4B$

$B = 8$

Now, when $x = 0$ and $B = 8$

$3(0)^2 - 16 = A(0 - 4) + 8(0)$

$-16 = -4A$

$A = 4$

Substitute $A = 4$ and $B = 8$ in

$\frac{3x^2-16}{x(x-4)} = \frac{A}{x} + \frac{B}{(x-4)}$

$\Rightarrow \frac{3x^2-16}{x(x-4)} = \frac{4}{x} + \frac{8}{(x-4)}$

3. $\frac{11x+12}{(2x+3)(x+2)(x-3)} = \frac{A}{2x+3} + \frac{B}{x+2} + \frac{C}{x-3}$

$11x + 12 = \frac{A(2x+3)(x+2)(x-3)}{2x+3} + \frac{B(2x+3)(x+2)(x+3)}{x+2} + \frac{C(2x+3)(x+2)(x-3)}{x-3}$

$11x + 12 = A(x + 2)(x - 3) + B(2x + 3)(x - 3) + C(2x + 3)(x + 2)$

When $x = -2$,

$11(-2) + 12 = B[2(-2) + 3](-2 - 3)$

$-22 + 12 = B(-4 + 3)(-5)$

$-10 = 5B$

When $x = 3$,

$11(3) + 12 = C[2(3) + 3](3 + 2)$

$33 + 12 = C(6 + 3)(5)$

$45 = C(9)(5)$

$45 = 45C$

$C = 1$

When $x = -\frac{3}{2}$,

$11\left(-\frac{3}{2}\right) + 12 = A\left(-\frac{3}{2} + 2\right)\left(-\frac{3}{2} - 3\right)$

$-\frac{9}{2} = A\left(\frac{1}{2}\right)\left(-\frac{9}{2}\right)$

$-\frac{9}{2} = A\left(-\frac{9}{4}\right)$

$-\frac{9}{2} = -\frac{9A}{4}$

$18A = 36$

$A = 2$

Substitute $A = 2$, $B = -2$ and $C = 1$ in $\frac{11x+12}{(2x+3)(x+2)(x-3)} = \frac{A}{2x+3} + \frac{B}{x+2} + \frac{C}{x-3}$

$\frac{11x+12}{(2x+3)(x+2)(x+3)} = \frac{2}{2x+3} + \frac{-2}{x+2} + \frac{1}{x-3}$

The partial fraction decomposition is;

$\frac{11x+12}{(2x+3)(x+2)(x+3)} = \frac{2}{2x+3} - \frac{2}{x+2} - \frac{1}{x-3}$

4. $\frac{4x^2+13x-9}{x^3-2x^2-3x}$

Factorize the denominator to obtain a quadratic expression;

$x^3 - 2x^2 - 3x = x(x^2 - 2x - 3)$

By factorization,

$x^2 - 2x - 3 = (x + 1)(x - 3)$

$x^3 - 2x^2 - 3x = x(x + 1)(x - 3)$

$\frac{4x^2+13x-9}{x(x^2-2x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3}$

$4x^2 + 13x - 9 = \frac{Ax(x+1)(x-3)}{x} + \frac{Bx(x+1)(x-3)}{x+1} + \frac{Cx(x+1)(x-3)}{x-3}$

$4x^2 + 13x - 9 = A(x + 1)(x - 3) + Bx(x - 3) + Cx(x + 1)$

When $x = 3$,
$4(3)^2 + 13(3) - 9 = 0 + 0 + C[3(3 + 1)]$
$36 + 39 - 9 = 12C$
$66 = 12C$
$C = \frac{11}{2}$

When $x = -1$,
$4(-1)^2 + 13(-1) - 9 = 0 + B[-1(-1 - 3)] + 0$
$4 - 13 - 9 = 4B$
$-18 = 4B$
$B = -\frac{9}{2}$

When $x = 0$,
$4(0)^2 + 13(0) - 9 = A(0 + 1)(0 - 3) + 0 + 0$
$-9 = A(1)(-3)$
$-9 = -3A$
$A = 3$

Substitute $A = 3$, $B = -\frac{9}{2}$ and $C = \frac{11}{2}$ in $\frac{4x^2+13x-9}{x(x^2-2x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3}$

$\Rightarrow \frac{4x^2+13x-9}{x(x^2-2x-3)} = \frac{3}{x} + \frac{-9/2}{x+1} + \frac{11/2}{x-3}$

The partial fraction decomposition is;
$\frac{4x^2+13x-9}{x(x^2-2x-3)} = \frac{3}{x} - \frac{9}{2(x+1)} + \frac{11}{2(x-3)}$

5. Express the rational function $\frac{7x+8}{(2x+3)(x-1)}$ in partial fractions.

Solution

$\frac{7x+8}{(2x+3)(x-1)} = \frac{A}{2x+3} + \frac{B}{x-1}$

Multiplying throughout by $(2x + 3)(x - 1)$,
$7x + 8 = A(x - 1) + B(2x + 3)$.

Substituting $x = 1$
$7 + 8 = B(2 + 3)$.
$15 = 5B$
$B = 3$.

Substituting $x = \frac{-3}{2}$
$7\left(\frac{-3}{2}\right) + 8 = A\left(\frac{-3}{2} - 1\right)$
$-\frac{5}{2} = -\frac{5}{2}A$
$A = 1$

$\frac{7x+8}{(2x+3)(x-1)} = \frac{1}{2x+3} + \frac{3}{x-1}$

.Exercises 6.7

A. Express in partial fractions:

1. $\frac{6}{(x+3)(x-3)}$
2. $\frac{x-29}{(x-4)(x+1)}$
3. $\frac{5x-12}{x^2-4x}$
4. $\frac{x}{(2+x)(2-x)}$

B. Resolve into partial fractions:

1. $\frac{x-1}{3x^2-11x+10}$
2. $\frac{x+34}{x^2-4x-12}$
3. $\frac{4x^2-15x-1}{(x-1)(x+2)(x-3)}$
4. $\frac{3-4x}{2+3x-2x^2}$
5. $\frac{x^2+19x+20}{x(x+2)(x-5)}$
6. $\frac{37-11x}{(x+1)(x^2-5x+6)}$

Challenge Problems

Resolve into partial fractions :

1. $\frac{4x^2-5x-15}{x^3-4x^2-5x}$
2. $\frac{4x^3+4x^2-4x+2}{2x^2-x-1}$
3. $\frac{3x+1}{(x+2)(x+1)(x-3)}$
4. $\frac{8x^2+13x+6}{(x+2)(2x+1)(3x+2)}$

Type II: A Partial fraction decomposition containing a repeated linear factor

A rational expression whose denominator contains a repeated linear factor is decomposed on the R.H.S. as shown below.

1. $\frac{Numerator}{x(x+e)^2} = \frac{A}{x} + \frac{B}{x+e} + \frac{C}{(x+e)^2}$
2. $\frac{Numerator}{x(x+e)^3} = \frac{A}{x} + \frac{B}{x+e} + \frac{C}{(x+e)^2} + \frac{D}{(x+e)^3}$,
3. $\frac{Numerator}{(x+e)^n} = \frac{A}{x+e} + \frac{B}{(x+e)^{n-1}} + \frac{C}{(x+e)^n}$,

where the numerators at the R.H.S. are constants to be found.

Worked Examples

Find the partial fraction decomposition of the following:

1. $\frac{2x+3}{(x-1)^2}$ 2. $\frac{x^2+10x-36}{x(x-3)^2}$

Solution

1. $\frac{2x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$

$\frac{2x+3}{(x-1)^2}(x-1)^2 = \frac{A}{x-1}(x-1)^2 + \frac{B}{(x-1)^2}(x-1)^2$

$2x + 3 = A(x - 1) + B$

When $x = 1$

$2(1) + 3 = A(1 - 1) + B$

$5 = 0 + B$

$B = 5$

Now, when $x = 0$, and $B = 5$

Substitute in $2x + 3 = A(x - 1) + B$

$2(0) + 3 = A(0 - 1) + 5$

$3 = -A + 5$

$A = 5 - 3$

$A = 2$

Alternatively;

From $2x + 3 = A(x - 1) + B$

$2x + 3 = Ax - A + B$

when $x = 0$, and $B = 5$

$2(0) + 3 = A(0) - A + 5$

$3 = -A + 5$

$A = 5 - 3$

$A = 2$

The partial fraction is: $\frac{2x+3}{(x-1)^2} = \frac{2}{x-1} + \frac{5}{(x-1)^2}$

2. $\frac{x^2+10x-36}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

Multiply through by lcd;

$\frac{x^2+10x-36}{x(x-3)^2}x(x-3)^2 = \frac{A}{x}x(x-3)^2 + \frac{B}{x-3}x(x-3)^2 + \frac{C}{(x-3)^2}x(x-3)^2$

$\Rightarrow x^2 + 10x - 36 = A(x - 3)^2 + Bx(x - 3) + Cx$

$x^2 + 10x - 36 = A(x^2 - 6x + 9) + B(x^2 - 3x) + Cx$

$x^2 + 10x - 36 = Ax^2 - 6Ax + 9A + Bx^2 - 3Bx + Cx$

$x^2 + 10x - 36 = Ax^2 + Bx^2 - 6Ax - 3Bx + Cx + 9A$

By regrouping;

$x^2 + 10x - 36 = (A + B)x^2 + (-6A - 3B + C)x + 9A$

Equating coefficients, the following is obtained;

$A + B = 1$……*(1)*

$-6A - 3B + C = 10$……*(2)*

$9A = -36$……*(3)*

From equation 3,

$9A = -36$

$A = -4$

Put $A = -4$ in eqn (1);

$-4 + B = 1$

$B = 1 + 4$

$B = 5$

Put $A = -4$, and $B = 5$ in eqn (2);

$-6(-4) - 3(5) + C = 10$

$24 - 15 + C = 10$
$9 + C = 10$
$C = 10 - 9 = 1$

The partial fraction is:
$\frac{x^2 + 10x - 36}{x(x-3)^2} = \frac{-4}{x} + \frac{5}{x-3} + \frac{1}{(x-3)^2}$

3. Express $\frac{3x^3 + x + 1}{(x-2)(x+1)^3}$ in partial fractions

Solution

$\frac{3x^3+x+1}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$

$\frac{3x^3+x+1}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$

$3x^3 + x + 1 = A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2)$

When $x = -1$;
$3(-1)^3 + (-1) + 1 = A(-1+1)^3 + B(-1-2)(-1+1)^2 + C(-1-2)(-1+1) + D(-1-2)$
$-3 - 1 + 1 = -3D$
$-3 = -3D$
$D = 1$

When $x = 2$;
$3(2)^3 + (2) + 1 = A(2+1)^3 + B(2-2)(2+1)^2 + C(2-2)(2+1) + D(2-2)$
$24 + 2 + 1 = A(3)^3$
$27 = 9A$
$A = 3$

Considering the R.H.S.
$A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2)$

Substitute $A = 3$ and $D = 1$
$3(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + 1(x-2)$

By expansion;
$3[x^3 + 3x^2 + 3x + 1] + B(x-2)(x^2 + 2x + 1) + C(x^2 + x - 2x - 2) + (x-2)$

$3[x^3 + 3x^2 + 3x + 1] + B(x^3 + 2x^2 + x - 2x^2 - 4x - 2) + C(x^2 + x - 2x - 2) + (x-2)$

$3[x^3 + 3x^2 + 3x + 1] + B(x^3 - 3x - 2) + C(x^2 - x - 2) + (x-2)$
$3x^3 + 9x^2 + 9x + 3 + Bx^3 - 3Bx - 2B + Cx^2 - Cx - 2C + x - 2$

$3x^3 + Bx^3 + 9x^2 + Cx^2 + 9x + x - 3Bx - Cx - 2B - 2C + 3 - 2$

$3x^3 + Bx^3 + 9x^2 + Cx^2 + 10x - 3Bx - Cx - 2B - 2C + 1$

$(3 + B)x^3 + (9 + C)x^2 + (10 - 3B - C)x - 2(B + C) + 1$

Comparing L. H. S and R. H. S;
That is comparing $3x^3 + x + 1$ to $(3 + B)x^3 + (9 + C)x^2 + (10 - 3B - C)x - 2(B + C) + 1$

Equating coefficients;
$3 + B = 3$……..…(1)
$9 + C = 0$……(2)
$10 - 3B - C = 1$……..(3)

From eqn (1);
$B = 0$;

From eqn (2);
$C = -9$

When $B = 0$ and $C = -9$, eqn (3) is satisfied.
$\frac{3x^3+x+1}{(x-2)(x+1)^3} = \frac{3}{x-2} + \frac{0}{x+1} + \frac{-9}{(x+1)^2} + \frac{1}{(x+1)^3}$

$\frac{3x^3+x+1}{(x-2)(x+1)^3} = \frac{3}{x-2} - \frac{9}{(x+1)^2} + \frac{1}{(x+1)^3}$

4. Express the rational function $\frac{9}{(x+1)^2(x-2)}$ in partial fractions.

Solution

$\frac{9}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$

$9 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$

When $x = -1$

$9 = A(-1+1)(-1-2) + B(-1-2) + C(-1+1)^2$

$9 = -3B$

$B = -3$

When $x = -2$

$9 = A(2+1)(2-2) + B(2-2) + C(2+1)^2$

$9 = C(3)^2$

$9 = 9C$

$C = 1$

When $B = -3$ and $C = 1$;

$9 = A(x+1)(x-2) + -3(x-2) + 1(x+1)^2$

$9 = A(x^2 - 2x + x - 2) - 3x + 6 + (x^2 + 2x + 2)$

$9 = A(x^2 - x - 2) - 3x + 6 + x^2 + 2x + 2$

$9 = Ax^2 - Ax - 2A - 3x + 6 + x^2 + 2x + 2$

$9 = Ax^2 + x^2 - Ax - 3x + 2x - 2A + 6 + 2$

$9 = Ax^2 + x^2 - Ax - x - 2A + 8$

$9 = x^2(A+1) - x(A+1) - 2A + 8$

$9 = -2A + 8$

$2A = 9 - 8$

$2A = 1$

$A = \frac{1}{2}$

The partial fraction decomposition is:

$\frac{9}{(x+1)^2(x-2)} = \frac{1/2}{x+1} + \frac{-3}{(x+1)^2} + \frac{1}{x-2}$

$\frac{9}{(x+1)^2(x-2)} = \frac{1}{2(x+1)} - \frac{3}{(x+1)^2} + \frac{1}{x-2}$

Exercises 6.8

Resolve into partial fractions.

1. $\frac{2x+3}{(x-1)^2}$
2. $\frac{10-x}{x^2+10x+25}$
3. $\frac{x^2-6}{(x+2)^2(2x-1)}$
4. $\frac{2x^2+x}{(x-1)^2(x+1)^2}$
5. $\frac{2x^2+7x}{x^2+6x+9}$
6. $\frac{4x^3+4x^2-4x+2}{2x^2-x-1}$

B. Express in partial fractions;

1. $\frac{x+1}{(x+1)^3}$
2. $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$
3. $\frac{5x^2+2}{(3x+1)(x+1)^2}$
4. $\frac{5x+4}{(x-1)(x+2)^2}$

Challenge Problem

1. Find the values of *A, B, C, D,* if

$\frac{x^3-10x^2+26x+3}{(x+3)(x-1)^3} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$

Type III: A Partial fraction decomposition containing an irreducible quadratic factor

A rational expression whose denominator contains an irreducible quadratic factor is decomposed on the R.H.S. as shown below;

1. $\frac{Numerator}{x(x^2+a)} = \frac{A}{x} + \frac{Bx+c}{x^2+a}$
2. $\frac{Numerator}{(x^2+a)(x+a)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{x+a}$

Worked Examples

1. Find the partial fraction decomposition of $\frac{9x^2-3x+8}{x^3+2x}$

Solution

$\frac{9x^2-3x+8}{x^3+2x}$

$x^3 + 2x = x(x^2+2)$ **(Factorize the denominator)**

$\frac{9x^2-3x+8}{x^3+2x} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$ ***(Decompose the expression)***

Multiply through by lcd, $x(x^2 + 2)$

$\Rightarrow 9x^2 - 3x + 8 = A(x^2 + 2) + (Bx + C)x$

$= Ax^2 + 2A + Bx^2 + Cx$

$= Ax^2 + Bx^2 + Cx + 2A$

$= (A + B)x^2 + Cx + 2A$

Now, equating coefficients;

$9x^2 - 3x + 8 = (A + B)x^2 + Cx + 2A$

$A + B = 9$ ……(1)

$C = -3$ ……..(2)

$2A = 8$……..(3)

From eqn (3),

$A = 4$

Put $A = 4$ into eqn (1);

$4 + B = 9$

$B = 9 - 4$

$B = 5$

Substitute $A = 4$, $B = 5$ and $C = -3$ *into*

$\frac{9x^2-3x+8}{x^3+2x} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$

$\Rightarrow \frac{9x^2-3x+8}{x^3+2x} = \frac{4}{x} + \frac{5x-3}{x^2+2}$

2. Resolve $\frac{4x^3-x^2+15x-29}{2x^3-x^2+8x-4}$ into partial fractions.

Solution

In $\frac{4x^3-x^2+15x-29}{2x^3-x^2+8x-4}$;

Degree of numerator is equal to degree of denominator.

Therefore, long division is required.

$$2x^3 - x^2 + 15x - 29 \;\overline{)\; 4x^3 - x^2 + 15x - 29 \;}\; 2$$

$$4x^3 - 2x^2 + 16x - 8$$

$$x^2 - x - 21$$

$= 2 + \frac{x^2-x-21}{2x^3-x^2+8x-4}$

$= x^2(2x - 1) + 4(2x - 1)$

$= (x^2 + 4)(2x - 1)$

By decomposition,

$\Rightarrow \frac{x^2-x-21}{2x^3-x^2+8x-4} = \frac{Ax+B}{x^2+4} + \frac{C}{2x-1}$

Multiply both sides by $(x^2 + 4)(2x - 1)$

$x^2 - x - 21 = (Ax + B)(2x - 1) + C(x^2 + 4)$

$= 2x(Ax + B) - 1(Ax + B) + C(x^2 + 4C)$

$= 2Ax^2 + 2Bx - Ax - B + Cx^2 + 4C$

$= 2Ax^2 + Cx^2 - Ax + 2Bx - B + 4C$

$x^2 - x - 21 = (2A + C)x^2 + (-A + 2B)x - B + 4C$

Now, equating coefficients in

$x^2 - x - 21 = (2A + C)x^2 + (-A + 2B)x - B + 4C$

$2A + C = 1$ ……(1)

$-A + 2B = -1$……(2)

$-B + 4C = -21$……(3)

From eqn (1);

$C = 1 - 2A$……(4)

Put eqn (4) into eqn (3);

$-B + 4(1 - 2A) = -21$

$-B + 4 - 8A = -21$……(5)

From eqn (2);

$-A = -1 - 2B$

$A = 1 + 2B$……..…(6)

Put eqn (6) into eqn (5);

$-B + 4 - 8(1 + 2B) = -21$

$-B + 4 - 8 - 16B = -21$

$-17B - 4 = -21$

$-17B = -17$

$B = 1$

Put B = 1 in eqn (2);

$-A + 2B = -1$

$-A + 2(1) = -1$

$-A + 2 = -1$

$-A = -1 - 2$

$-A = -3$

$A = 3$

Put $A = 3$ into eqn (1);

$2(3) + C = 1$

$6 + C = 1$

$C = 1 - 6$

$C = -5$

Substitute $A = 3$, $B = 1$ and $C = -5$ into

$$\Rightarrow \frac{x^2-x-21}{2x^3-x^2+8x-4} = \frac{Ax+B}{x^2+4} + \frac{C}{2x-1}$$

$$= \frac{x^2-x-21}{2x^3-x^2+8x-4} = \frac{3x+1}{x^2+4} + \frac{-5}{2x-1}$$

The partial fraction of the expression is;

$$\frac{4x^3-x^2+15x-29}{2x^3-x^2+8x-4} = 2 + \frac{3x+1}{x^2+4} - \frac{5}{2x-1}$$

3. Express the rational function $\frac{3x^2+9}{(x-5)(x^2+2x+7)}$ in partial fractions.

Solution

$$\frac{3x^2+9}{(x-5)(x^2+2x+7)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+2x+7}$$

Multiplying throughout by:

$(x - 5)(x^2 + 2x + 7)$,

$3x^2 + 9 = A(x^2 + 2x + 7) + (Bx + C)(x - 5)$.

Substitute $x = 5$

$3 \times 5^2 + 9 = A(5^2 + 2 \times 5 + 7)$.

$84 = 42A$

$A = 2$.

Equating coefficients;

$3x^2 + 9 = A(x^2 + 2x + 7) + (Bx + C)(x - 5)$.

$3x^2 + 9 = Ax^2 + 2Ax + 7A + Bx^2 + Cx - 5Bx - 5C$

$3x^2 + 9 = Ax^2 + Bx^2 + 2Ax - 5Bx + Cx + 7A - 5C$

$3x^2 + 9 = (A + B)x^2 + (2A - 5B + C)x + 7A - 5C$

$A + B = 3$……(1)

$2A - 5B + C = 0$……(2)

$7A - 5C = 9$……(3)

From eqn (1);

$A + B = 3$

$2 + B = 3$ (But $A = 2$)

$B = 3 - 2$

$B = 1$

From equation (2);

Substitute $A = 2$ and $B = 1$

$2(2) - 5(1) + C = 0$

$4 - 5 + C = 0$

$-1 + C = 0$

$C = 1$

Substitute $A = 2$, $B = 1$ and $C = 1$

$$\frac{3x^2+9}{(x-5)(x^2+2x+7)} = \frac{2}{x-5} + \frac{x+1}{x^2+2x+7}$$

Exercises 6.9

A. Resolve into partial fractions:

1. $\frac{5x^2-4}{x^2(x+2)}$
2. $\frac{x^2+x-6}{(x^2+1)(x-1)}$
3. $\frac{19x^2+50x-25}{3x^3-5x^2}$
4. $\frac{5x^2-10x+11}{(x-3)(x^2+4)}$
5. $\frac{3x^2-2x+5}{(x-1)(x^2+5)}$
6. . $\frac{4x^2+x+1}{x(x^2-1)}$
7. $\frac{2x^2-x+3}{(x+1)(x^2+2)}$
8. $\frac{2x^2-x-1}{(x-3)(x^2+1)}$

B. Express in partial fractions;

1. $\frac{3}{x^2(x+2)}$
2. $\frac{6-x}{(1-x)(4+x^2)}$
3. $\frac{5x+2}{(x+1)(x^2-4)}$

4. $\frac{20x+84}{(x+5)(x^2-9)}$ 5. $\frac{4}{(x+1)(2x^2+x+3)}$ 6. $\frac{3+2x}{(2-x)(3+x^2)}$

Type IV: A Partial fraction decomposition containing a repeated quadratic factor

A rational expression whose denominator contains a repeated quadratic factor is decomposed on the R.H.S. as shown below;

1. $\frac{Numerator}{(x^2+a)^2} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{(x^2+a)^2}$

Worked Examples

Find the partial fraction decomposition of $\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}$

Solution

The degree of the numerator is 3 and the degree of the denominator is 4. Long division is not required.

$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

Multiply both sides by lcd, $(x^2 + 1)^2$

$\Rightarrow 5x^3 - 3x^2 + 7x - 3$

$= (Ax + B)(x^2 + 1) + Cx + D$

$= Ax(x^2 + 1) + B(x^2 + 1) + Cx + D$

$= Ax^3 + Ax + Bx^2 + B + Cx + D$

$= Ax^3 + Bx^2 + (A + C)x + B + D$

Comparing coefficients,

$5x^3 - 3x^2 + 7x - 3 = Ax^3 + Bx^2 + (A + C)x + B + D$

$A = 5$

$B = -3$

$A + C = 7$ ……(1)

$B + D = -3$……..(2)

Put $A = 5$ in eqn (1);

$5 + C = 7$

$C = 7 - 5$

$C = 2$

Put $B = -3$ in eqn (2)

$-3 + D = -3$

$D = -3 + 3$

$D = 0$

Substitute $A = 5$, $B = -3$, $C = 2$ and $D = 0$ in

$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2}$

Exercises 6.10

Resolve into partial fractions.

1. $\frac{4x^3-x^2+4x+2}{(x^2+1)^2}$ 2. $\frac{3x^3+13x-1}{(x^2+4)^2}$